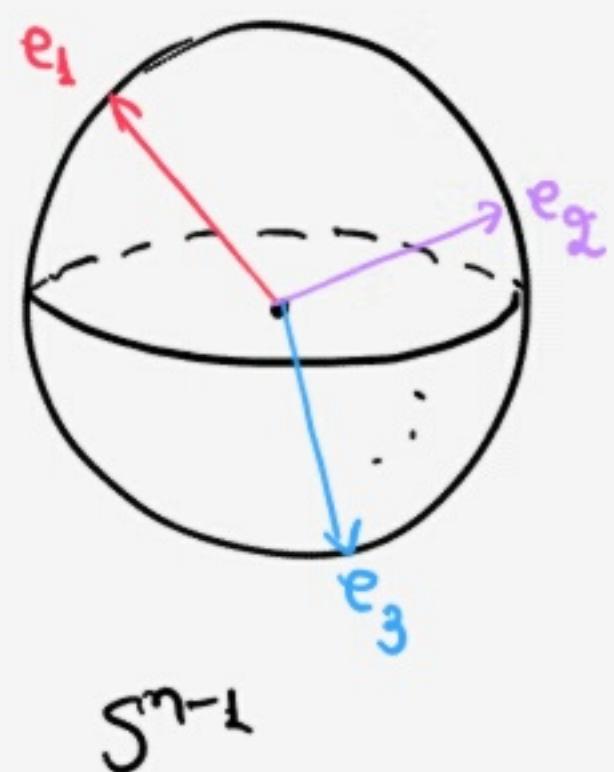


## Lecture 19 (21/2/2014)

Def: A Kakeya set in  $\mathbb{R}^n$  is a compact subset of  $\mathbb{R}^n$  containing a unit line segment in each direction.

i.e.,  $\forall e \in S^{n-1}$ , there exists a line segment in  $K$  with length 1 and direction  $e$ .



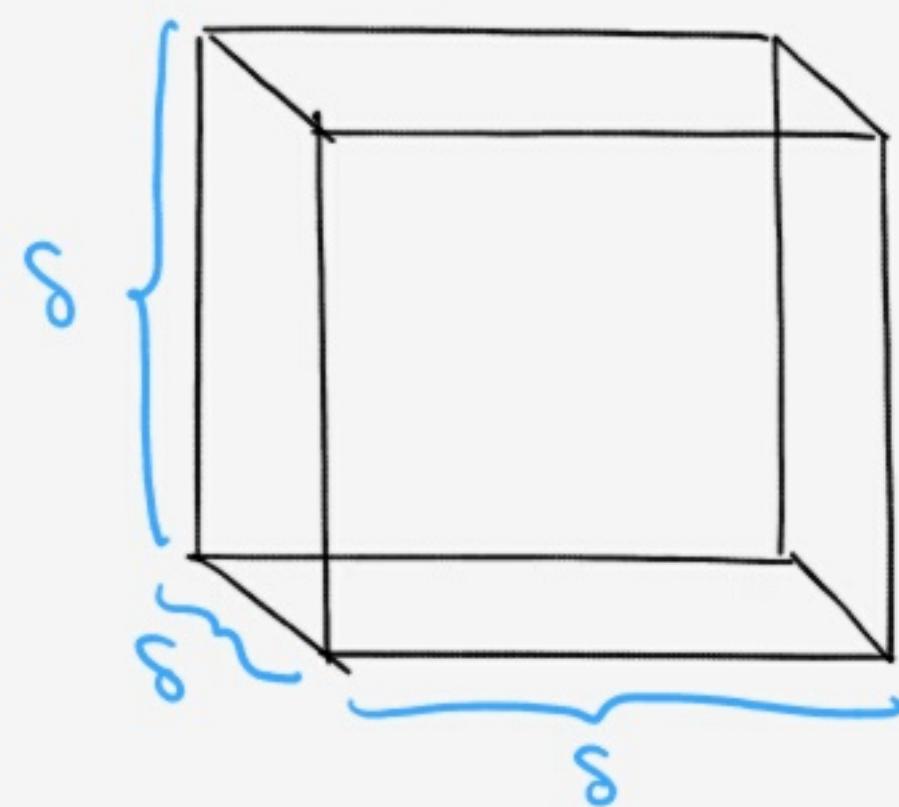
a Kakeya set in  $\mathbb{R}^n$ .

Kakeya set conjecture: Any Kakeya set in  $\mathbb{R}^n$  has Hausdorff dimension  $n$ .

So far, the conjecture has been fully solved only in  $\mathbb{R}^2$ , and there is partial progress for  $n \geq 3$ : we know that every Kakeya set has Hausdorff dimension  $\geq \frac{n+2}{2}$ .

It is in general easier, when working on Kakeya sets, to work with another notion of dimension: the upper Minkowski dimension.

Def: A  $\delta$ -cube in  $\mathbb{R}^n$  is a cube in  $\mathbb{R}^n$  of side  $\delta$ .



Notation: Let  $E \subseteq \mathbb{R}^n$ . We denote by  $E^\delta$  the  $\delta$ -neighbourhood of  $E$  in  $\mathbb{R}^n$ .

Def: Let  $E \subseteq \mathbb{R}^n$  be bounded. We define the upper Minkowski dimension of  $E$  as

$$\dim_M E := \limsup_{\delta \rightarrow 0} \frac{\log N(\delta)}{\log\left(\frac{1}{\delta}\right)},$$

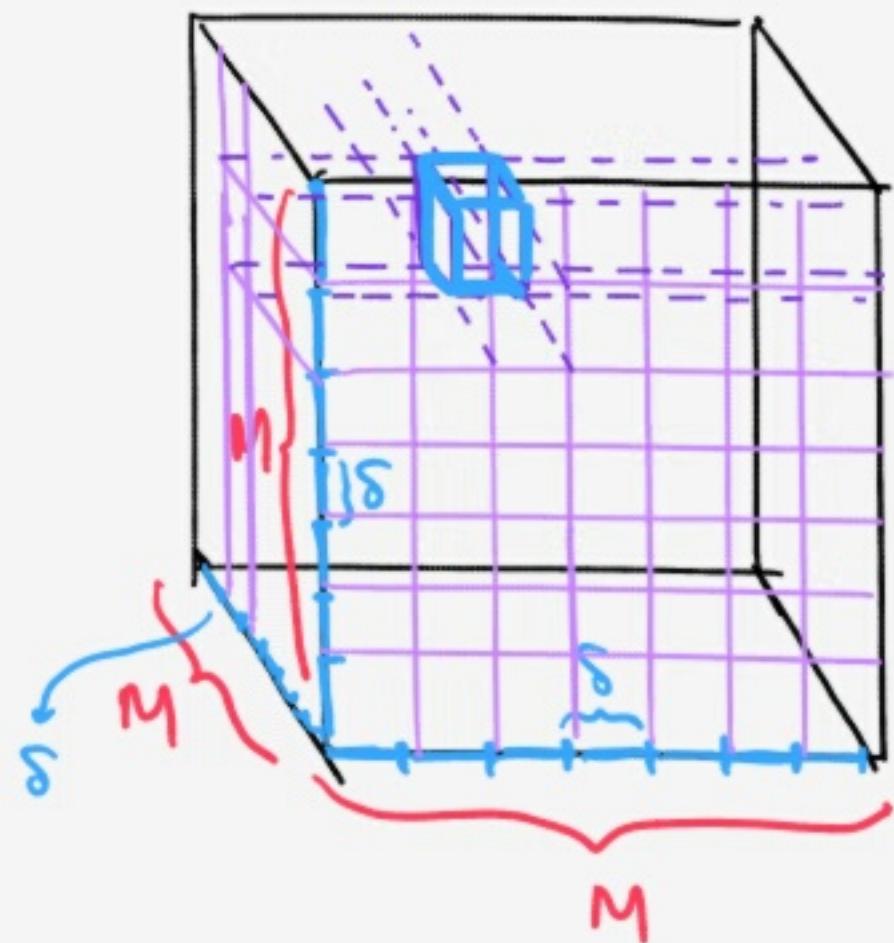
where  $N(\delta)$  is the smallest number of  $\delta$ -cubes in  $\mathbb{R}^n$  needed to cover  $E^\delta$ .

→ How many  $\delta$ -cubes in  $\mathbb{R}^n$  are needed to cover an  $M$ -cube in  $\mathbb{R}^n$ , for  $\delta$  small?

Answer:  $\left(\frac{M}{\delta}\right)^n$ .

Indeed, there are two ways to see this:

1st:

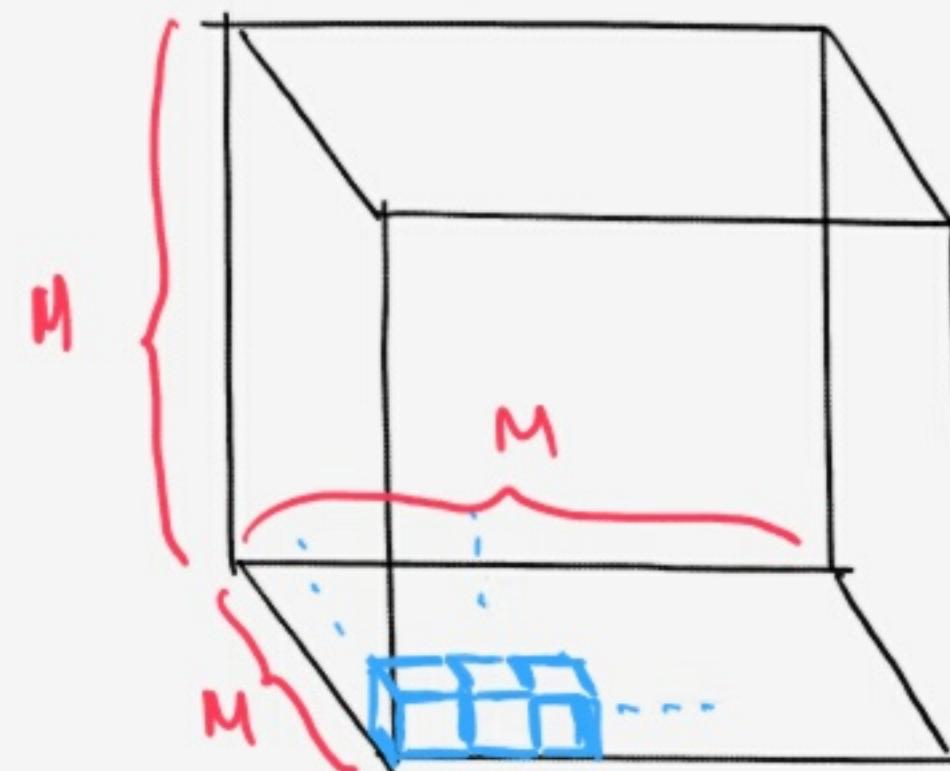


To cover the  $M$ -cube by  $\delta$ -cubes in an optimal way, we divide each of the  $n$  "axes" of the  $M$ -cube in equal intervals of length  $\delta$ , i.e. in  $\frac{M}{\delta}$  intervals.

This way we form a grid of  $\underbrace{\frac{M}{\delta} \dots \frac{M}{\delta}}_{n \text{ times}} = \left(\frac{M}{\delta}\right)^n$

$\delta$ -cubes inside the  $M$ -cube, which cover the  $M$ -cube.

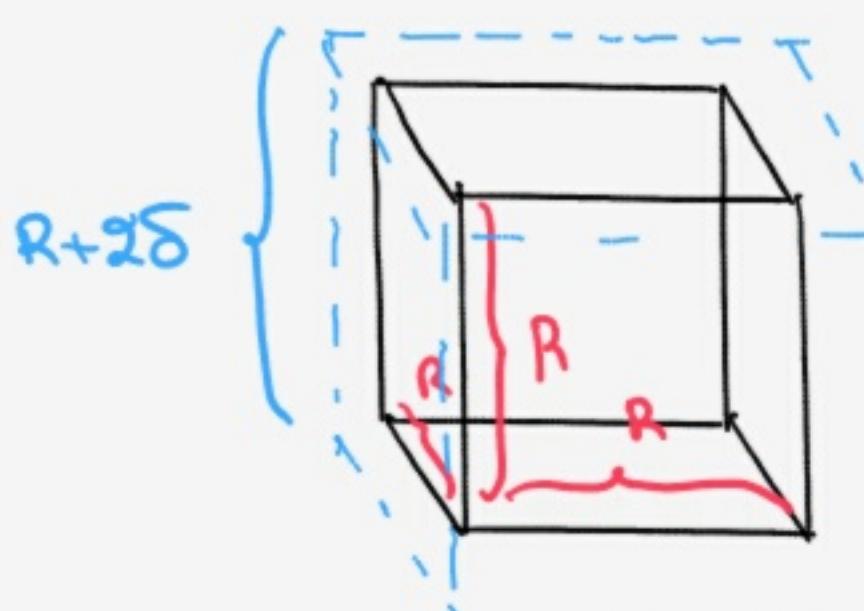
2nd:



We want to divide a cube of volume  $M^n$  in cubes of volume  $\delta^n$ . For that, we need

$$\frac{M^n}{\delta^n} \text{ } \delta\text{-cubes.}$$

## Properties of the upper Minkowski dimension:

- If  $A \subseteq B$  are bounded subsets of  $\mathbb{R}^n$ , then  $\dim_M A \leq \dim_M B$ .  
 The reason is that  $A^\delta \subseteq B^\delta$  for all  $\delta > 0$ , so we need more  $\delta$ -cubes to cover  $B^\delta$  than to cover  $A^\delta$ ; thus,  $N(\delta)$  is larger for  $B^\delta$  than for  $A^\delta$ , for all  $\delta > 0$ .
- Any cube in  $\mathbb{R}^n$  has upper Minkowski dimension  $n$ .  
 Indeed: Consider an  $R$ -cube, for some  $R > 0$ .
  - $\dim_M (R\text{-cube}) \leq n$ :  


It is clear that  $(R\text{-cube})^\delta$  is contained in an  $(R+2\delta)$ -cube, which, as we have explained, needs  $\frac{(R+2\delta)^n}{\delta^n}$   $\delta$ -cubes to be covered. So,  $N(\delta) \leq \frac{(R+2\delta)^n}{\delta^n}$ ,

thus, for all small  $\delta > 0$ ,

$$\frac{\log N(\delta)}{\log \left(\frac{1}{\delta}\right)} \leq \frac{\log (R+\delta)^n}{\log \left(\frac{1}{\delta}\right)} + \frac{\log \left(\frac{1}{\delta}\right)^n}{\log \left(\frac{1}{\delta}\right)} = n \cdot \underbrace{\frac{\log (R+\delta)}{\log \left(\frac{1}{\delta}\right)}}_{\downarrow \delta \rightarrow 0} + n \xrightarrow[\delta \rightarrow 0]{ } n, \text{ so}$$

$$\dim_N(R\text{-cube}) \leq n.$$

- $\dim_N(R\text{-cube}) \geq n$ :  $(R\text{-cube})^\delta \supseteq (R\text{-cube})$ , so  $N(\delta)$ , the number of  $\delta$ -cubes needed to cover  $(R\text{-cube})^\delta$ , is  $\geq$  the number of  $\delta$ -cubes needed to cover the  $(R\text{-cube})$ ,

$$\text{i.e. } \geq \frac{R^n}{\delta^n} \cdot S_0, \quad \frac{\log N(\delta)}{\log \left(\frac{1}{\delta}\right)} \geq \frac{\log R^n + \log \left(\frac{1}{\delta}\right)^n}{\log \left(\frac{1}{\delta}\right)} = n \cdot \underbrace{\frac{\log R}{\log \left(\frac{1}{\delta}\right)}}_{\downarrow \delta \rightarrow 0} + n \xrightarrow[\delta \rightarrow 0]{ } n,$$

$$\text{so } \dim_N(R\text{-cube}) \geq n.$$

- Let  $E \subseteq \mathbb{R}^n$  bounded. We have that  $\dim_N E \leq n$ .

Indeed, since  $E$  is bounded, it is contained in an  $R$ -cube in  $\mathbb{R}^n$ , for some  $R > 0$ .

So,  $\dim_M E \leq \dim_M (R\text{-cube}) = n$ .

## Lecture 20 (4/12/2014)

for any  $E \subseteq \mathbb{R}^n$  bounded,  $\dim_N E \geq \dim_H E$  (exercises).

Therefore, in principle showing that the Hausdorff dimension of a Kakeya set in  $\mathbb{R}^n$  is  $n$  is harder than showing that its upper Minkowski dimension is  $n$ . In fact, there has been some extra progress on the following weaker version of the Kakeya set conjecture (still very small though):

Kakeya set conjecture (weak version): Any Kakeya set in  $\mathbb{R}^n$  has upper Minkowski dimension  $n$ .

The aim of this lecture is to show that if, for a Kakeya set  $K$  in  $\mathbb{R}^n$ ,  $K^\delta$  is appropriately large, for all  $\delta > 0$ , then  $\dim_M K = n$ .

We start with the following.

Observation: Let  $E \subseteq \mathbb{R}^n$ , bounded. Then,

$$N(\delta) \geq \frac{|E^\delta|}{\delta^n}, \text{ for all } \delta > 0.$$

(Here,  $N(\delta)$  is the number of  $\delta$ -cubes in  $\mathbb{R}^n$  needed to cover  $E^\delta$ ).

Proof: Let  $\delta > 0$ , and  $\mathcal{N}_\delta$  be a cover of  $E^\delta$  by  $N(\delta)$   $\delta$ -cubes.

$$\text{Then, } E^\delta \subseteq \bigcup_{Q \in \mathcal{N}_\delta} Q \Rightarrow |E^\delta| \leq \sum_{Q \in \mathcal{N}_\delta} |Q| = \#\mathcal{N}_\delta \cdot \delta^n = N(\delta) \cdot \delta^n,$$

$$\text{so } N(\delta) \geq \frac{|E^\delta|}{\delta^n}.$$

Using this observation, we will show that, if  $E \subseteq \mathbb{R}^n$  is such that  $E^\delta$  is appropriately large, then  $\dim_N E$  is large as well:

Prop: Let  $E \subseteq \mathbb{R}^n$  bounded. If  $|E^\delta| \geq \delta^{n-\alpha}$  for all  $\delta > 0$ , then  $\dim_N E \geq \alpha$ .

Proof: for all  $\delta > 0$ ,  $N(\delta) \geq \frac{|E^\delta|}{\delta^n} \geq \frac{\delta^{n-\alpha}}{\delta^n} = \left(\frac{1}{\delta}\right)^\alpha$ ,

$$\text{so } \frac{\log N(\delta)}{\log\left(\frac{1}{\delta}\right)} \geq \frac{\log\left(\frac{1}{\delta}\right)^\alpha}{\log\left(\frac{1}{\delta}\right)} = \alpha,$$

$$\text{so } \dim_N E \geq \alpha.$$

This is good; but we would like to get the same result by allowing  $\epsilon^\delta$  to be a little smaller. For example, what if  $|\epsilon^\delta| \geq \frac{\delta^{n-\alpha}}{100}$  for all  $\delta > 0$ ,

or, even better, if  $|\epsilon^\delta| \geq \frac{\delta^{n-\alpha}}{\eta^n}$  for all  $\delta > 0$ ? Would  $\dim_N \epsilon$  still be  $\geq \alpha$ ? The answer is yes:

Prop: Let  $\epsilon \subseteq \mathbb{R}^n$  bounded. If  $|\epsilon^\delta| \geq c_n \cdot \delta^{n-\alpha}$  for all  $\delta > 0$ , for a constant  $c_n > 0$  depending only on  $n$ , then  $\dim_N \epsilon \geq \alpha$ .

Proof: For all  $\delta > 0$ ,  $N(\delta) \geq \frac{|\epsilon^\delta|}{\delta^n} \geq \frac{c_n \cdot \delta^{n-\alpha}}{\delta^n} = c_n \cdot \left(\frac{1}{\delta}\right)^\alpha$ ,

$$\text{so } \frac{\log N(\delta)}{\log \left(\frac{1}{\delta}\right)} \geq \frac{\log c_n}{\log \left(\frac{1}{\delta}\right)} + \frac{\log \left(\frac{1}{\delta}\right)^\alpha}{\log \left(\frac{1}{\delta}\right)} = \underbrace{\frac{\log c_n}{\log \left(\frac{1}{\delta}\right)}}_{\delta \rightarrow 0} + \alpha \xrightarrow[\delta \rightarrow 0]{} \alpha,$$

so  $\dim_M E \geq a$ .

And how about we allow  $E^\delta$  to be even smaller, such as  $\frac{\delta^{n-a}}{\log(\frac{1}{\delta}) n^n}$  ?

Prop: Let  $E \subseteq \mathbb{R}^n$ , bounded. If  $|E^\delta| \geq c_{n,1} \cdot \frac{1}{\left[\log\left(\frac{1}{\delta}\right)\right]^{c_{n,2}}} \cdot \delta^{n-a}$  for all  $\delta > 0$ ,

for constants  $c_{n,1}, c_{n,2} > 0$  depending only on  $n$ , then  $\dim_M E \geq a$ .

Proof: for all  $\delta > 0$ ,  $N(\delta) \geq \frac{|E^\delta|}{\delta^n} \geq c_{n,1} \cdot \left(\log \frac{1}{\delta}\right)^{-c_{n,2}} \cdot \left(\frac{1}{\delta}\right)^a$ , so

$$\frac{\log N(\delta)}{\log\left(\frac{1}{\delta}\right)} \geq \underbrace{\frac{\log c_{n,1}}{\log\left(\frac{1}{\delta}\right)}}_{\delta \rightarrow 0} - c_{n,2} \cdot \underbrace{\frac{\log\left(\log \frac{1}{\delta}\right)}{\log\frac{1}{\delta}}}_{\delta \rightarrow 0} + a \xrightarrow[\delta \rightarrow 0]{} 0,$$

$$\text{so } \dim_M E \geq a.$$

The above implies that, when studying the upper Minkowski dimension, constants that depend only on  $n$  and powers of  $\log \frac{1}{\delta}$  can be ignored in multiplication, i.e. can be seen as the number 1. This is reflected in the notation below:

Notation: Suppose that we work in  $\mathbb{R}^n$ . For positive quantities  $A, B$ , we write

$A \lesssim B$  if there exists a constant  $C_n$ , depending only on  $n$ , s.t.

$$A \leq C_n \cdot B.$$

In other words:  $C_n \leq 1$ , for any constant  $c_n > 0$  depending only on  $n$ .

ex:  $n^2 \lesssim 1, n! \lesssim 1, n^n \lesssim 1, 1 \lesssim C_n$

Moreover, we write  $A \gtrsim B$  if  $B \lesssim A$ , i.e.  $A \gtrsim c_n \cdot B$ , for some  $c_n > 0$

depending only on  $n$ ,

and  $A \sim B$  if  $A \lesssim B$  and  $A \gtrsim B$ , i.e.  $c_{n,1} B \leq A \leq c_{n,2} B$ ,

for some  $c_{n,1}, c_{n,2} > 0$

depending only on  $n$ .

- ex: •  $c_n \sim 1$ , for any constant  $c_n$  depending only on  $n$ .
- $|B(x, r)| \sim r^n$ .

- 



$$|E| \sim r^n.$$

→ We have shown that, if  $|E^\delta| \gtrsim \delta^{n-a}$  for all  $\delta > 0$ , then  
 $\dim_N E \geq a$ .

Notation: Suppose that we work in  $\mathbb{R}^n$ . For positive quantities  $A(\delta)$  and  $B(\delta)$ , which may depend on  $\delta$ , we write  $A(\delta) \lesssim B(\delta)$  if

$A(\delta) \leq C_{n,1} \cdot \log\left(\frac{1}{\delta}\right)^{C_{n,2}} \cdot B(\delta)$  for all  $\delta > 0$ , for constants  $C_{n,1}, C_{n,2} > 0$  that depend only on  $n$  (and not on  $\delta$ ).

In other words:  $C_{n,1} \cdot \left(\log\frac{1}{\delta}\right)^{C_{n,2}} \lesssim 1$ , for any  $C_{n,1}, C_{n,2} > 0$  that depend only on  $n$ .

ex:  $n^{100} \cdot \left(\log\frac{1}{\delta}\right)^n \lesssim 1$ ,  $n^{100} \lesssim 1$ ,  $1 \lesssim \left(\log\frac{1}{\delta}\right)^{10}$ .

Moreover, we write  $A(\delta) \gtrsim B(\delta)$  if  $B(\delta) \lesssim A(\delta)$ , i.e. if

$$A(\delta) \geq c_{n,1} \cdot \frac{1}{\left(\log \frac{1}{\delta}\right)^{c_{n,2}}} \cdot B(\delta) \text{ for all } \delta > 0,$$

for constants  $c_{n,1}, c_{n,2} > 0$  depending only on  $n$ ,

and  $A(\delta) \approx B(\delta)$  if  $A \lesssim B$  and  $A \gtrsim B$ .

→ We have shown that, if  $|E^\delta| \gtrsim \delta^{n-\alpha}$  for all  $\delta > 0$ , then  $\dim_N E \geq \alpha$ .

Corollary: Let  $K$  be a Kakeya set in  $\mathbb{R}^n$ . If  $|K^\delta| \gtrsim 1$  for all  $\delta > 0$ , then  $\dim_N K = n$ .

Proof: •  $\dim_M K \leq n$ , as  $K \subseteq \mathbb{R}^n$ .

•  $\dim_M K \geq n$ :  $|K^\delta| \gtrsim \underbrace{\delta^n}_{\substack{\parallel \\ 1}}$  for all  $\delta > 0$ , so  $\dim_M K \geq n$ .

Conjecture: Let  $K$  be a Kakeya set in  $\mathbb{R}^n$ . Then,  $|K^\delta| \gtrsim 1$  for all  $\delta > 0$ .

From the above discussion, this conjecture implies that any Kakeya set in  $\mathbb{R}^n$  has upper Minkowski dimension  $n$ .

## Lecture 21 (9/12/2014)

Let  $K$  be a Kakeya set in  $\mathbb{R}^n$ . In the previous lecture, we explained that, if

$$|K^\delta| \gtrsim L \text{ for all } \delta > 0, \quad (*)$$

then  $\dim_N K = n$ .

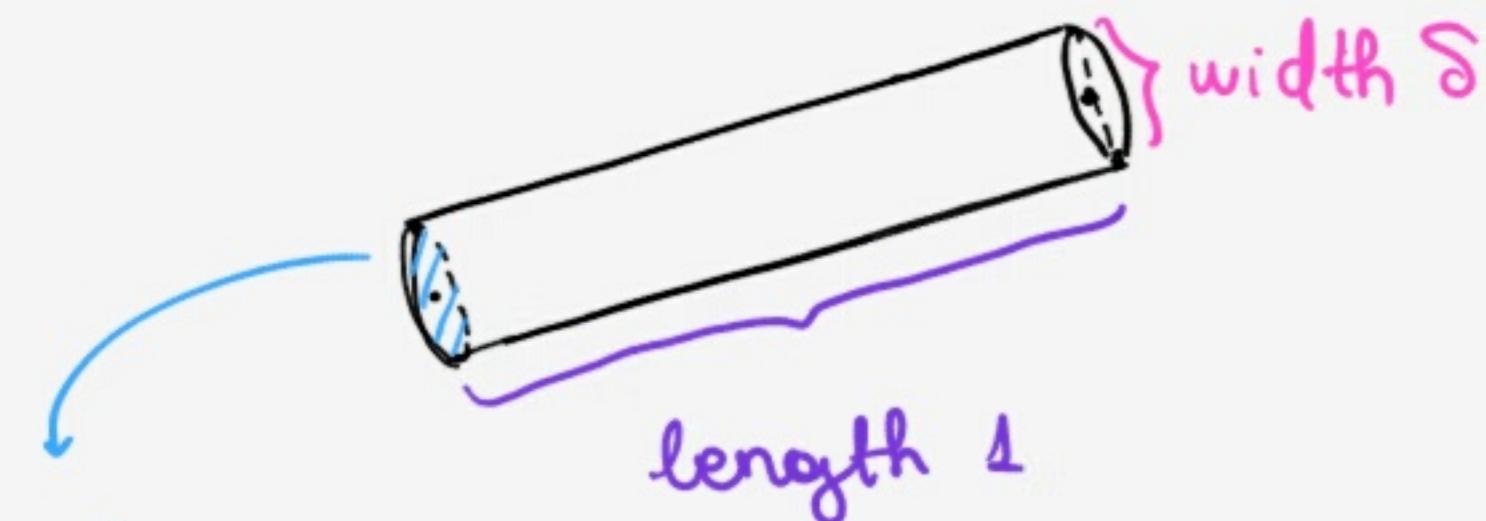
In this lecture, we will show that  $(*)$  is implied by another conjecture, stronger than the Kakeya set conjecture: the maximal Kakeya operator conjecture.

first, we need to find some structure in  $K^\delta$ , for all  $\delta > 0$ .

Indeed, let  $\delta > 0$ .  $K$  contains a unit line segment in each direction, so

$K^\delta$  contains a tube of radius  $\delta$  and length 1 in each direction,  
so  $K^\delta$  contains a tube of width  $\delta$  and length 1 in each direction.

Def: A  $\delta$ -tube in  $\mathbb{R}^n$  is the  $\frac{\delta}{2}$ -neighbourhood of a unit line segment in  $\mathbb{R}^n$ :



cross-section:

an  $(n-1)$ -dim ball

of radius  $\frac{\delta}{2}$ .

By the discussion above,  $K^\delta$  contains a  $\delta$ -tube in each direction.

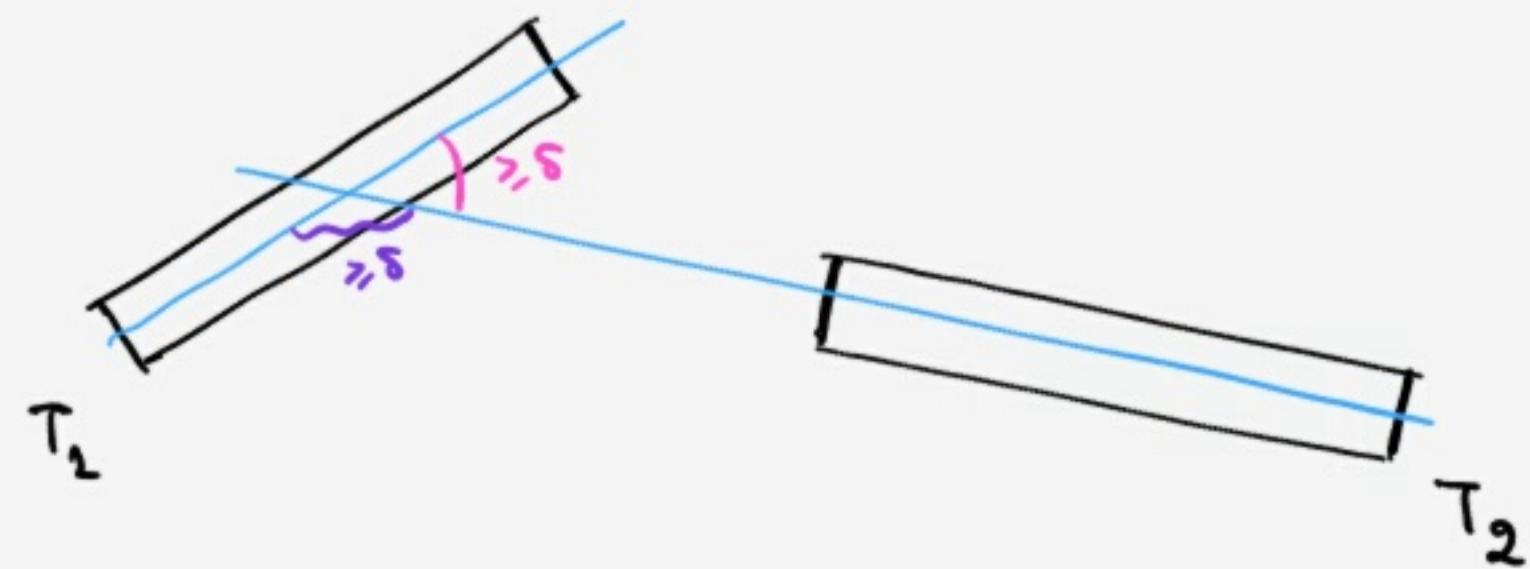
for many reasons, we are interested in  $\delta$ -tubes inside  $K^\delta$  with angle  $\geq \delta$  between them.

One of these reasons is that, if two  $\delta$ -tubes with angle  $< \delta$  intersect, then they

overlap a lot, so they are essentially the same tube. So:

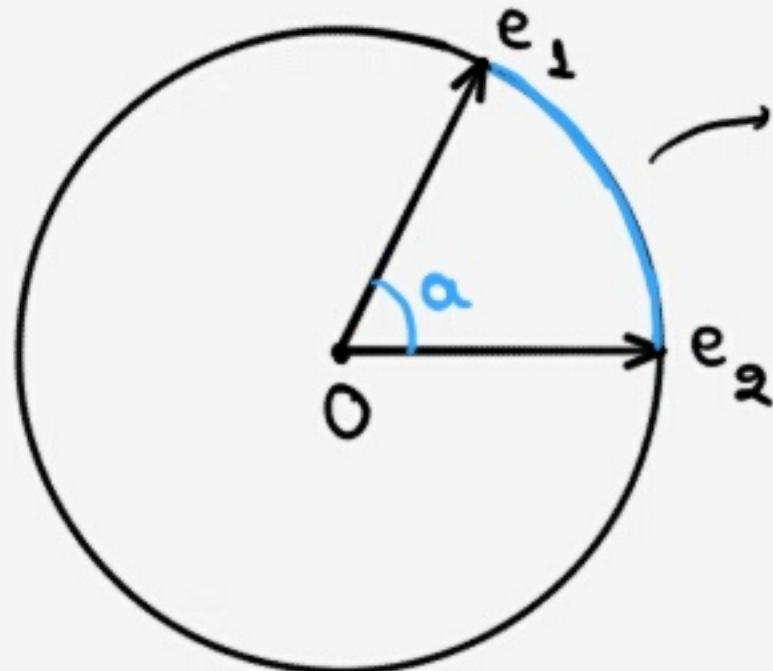
Notation: for any tube  $T$ , we denote by  $\text{dir}(T)$  the direction of  $T$ , with  $\text{dir}(T)$  in the upper semisphere,  $S_u^{n-1}$ .

Def: We say that the tubes  $T_1, T_2$  in  $\mathbb{R}^n$  are  $\delta$ -separated if the angle between their axes is  $\geq \delta$ , i.e. if  $\angle(\text{dir}(T_1), \text{dir}(T_2)) \geq \delta$   
and  $\angle(-\text{dir}(T_1), \text{dir}(T_2)) \geq \delta$ .



We will now find the cardinality of a maximal  $\delta$ -separated family of  $\delta$ -tubes contained in  $K^\delta$ .

Obs.:



$S^{n-1}$

$$\text{length} = c_n \alpha$$

If angle  $(e_1, e_2) = \alpha$ , then the distance between  $e_1$  and  $e_2$  on  $S^{n-1}$  is  $c_n \alpha$ , for  $c_n$  a constant depending only on  $n$ .

Def: A  $\delta$ -cap on  $S^{n-1}$ , centered at  $x \in S^{n-1}$ , is the set

$\{y \in S^{n-1} : \text{the distance of } y \text{ from } x \text{ on } S^{n-1} \text{ is } \leq \delta\}$ .

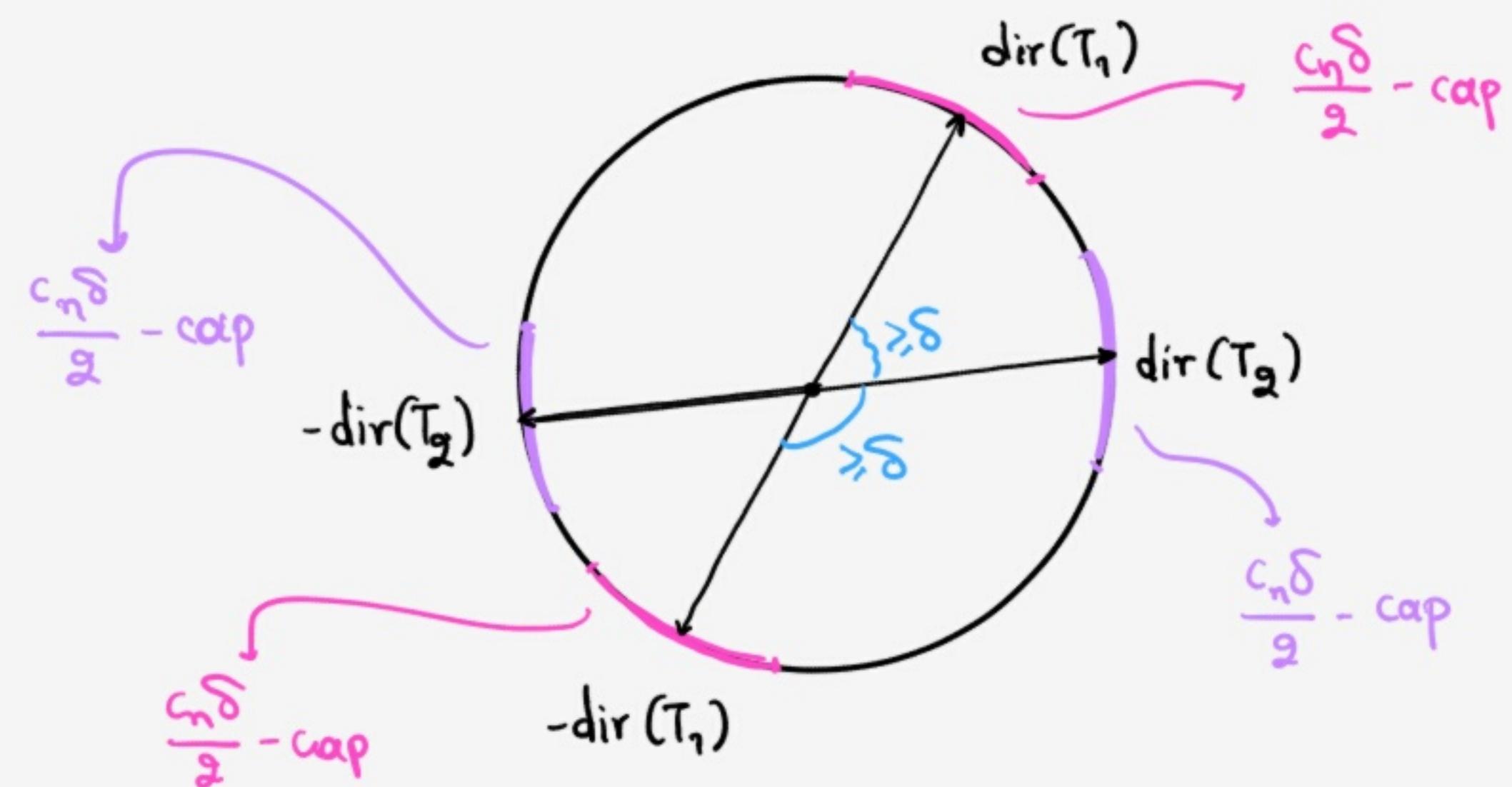
Now, by the observation above, two tubes  $T_1, T_2$  in  $\mathbb{R}^n$  are  $\delta$ -separated.

$\iff \text{angle}(\text{dir}(T_1), \text{dir}(T_2)) \geq \delta$  and

$\text{angle}(-\text{dir}(T_1), \text{dir}(T_2)) \geq \delta$

$\iff$  the  $\frac{c_n \delta}{2}$ -caps centered at  $\text{dir}(T_1), -\text{dir}(T_1), \text{dir}(T_2), -\text{dir}(T_2)$

are pairwise disjoint:



Therefore, a family  $\Pi$  of  $\delta$ -separated tubes in  $\mathbb{R}^n$  can have as many tubes in it as the number of pairwise disjoint antipodal pairs of  $\frac{c_n \delta}{2}$ -caps we can fit in  $S^{n-1}$ , i.e.

$$\leq \frac{(\text{surface area of } S^{n-1})}{(\text{surface area of the union of two antipodal } \frac{c_n \delta}{2} - \text{caps in } S^{n-1})} \sim$$

$$\sim \frac{\frac{1}{2} \left( \frac{c_n \delta}{2} \right)^{n-1}}{2 \cdot \left( \frac{c_n \delta}{2} \right)^{n-1}} \sim \frac{1}{\delta^{n-1}}, \quad \text{because } (\text{surface area of an } r\text{-cap on } S^{n-1}) = c_n r^{n-1} \sim r^{n-1},$$

and thus

$$(\text{surface area of a } \frac{c_n \delta}{2} - \text{cap on } S^{n-1}) \sim \delta^{n-1},$$

and       $(\text{surface area of } S(x, r) \text{ in } \mathbb{R}^n) = c_n r^{n-1},$   
 and thus

$$(\text{surface area of } S^{n-1}) \sim 1.$$

Therefore:

Any family  $\mathbb{T}$  of  $\delta$ -separated tubes in  $\mathbb{R}^n$  is s.t.  $\#\mathbb{T} \lesssim \frac{1}{\delta^{n-1}}$ .

In fact:

Any maximal family  $\mathbb{T}$  of  $\delta$ -separated tubes in  $\mathbb{R}^n$  is s.t.  $\#\mathbb{T} \sim \frac{1}{\delta^{n-1}}$ ,

because  $\mathbb{T}$  is maximal if the tubes in  $\mathbb{T}$  correspond to a maximal set of pairwise disjoint pairs of antipodal  $\frac{c_n \delta}{2}$ -caps as above, which has cardinality  $\sim \frac{1}{\delta^{n-1}}$

(because this is how many caps like that we need to cover the whole sphere).

In other words,  $\sim \frac{1}{\delta^{n-1}}$  is the number of  $c_n \delta$ -separated directions on  $S_u^{n-1}$ .

Therefore, the following holds:

for all  $\delta > 0$ ,  $K^\delta$  contains a family  $\mathbb{T}$  of  $\delta$ -separated  $\delta$ -tubes in  $\mathbb{R}^n$ , with  $\#\mathbb{T} \sim \frac{1}{\delta^{n-1}}$ :

Indeed, we have explained that  $K^\delta$  contains a  $\delta$ -tube in each direction;

therefore,  $K^\delta$  contains a  $\delta$ -tube in each direction in a maximal set of  $c_n \delta$ -separated directions in  $S_u^{n-1}$ , which has cardinality  $\sim \frac{1}{\delta^{n-1}}$ .

We have now revealed some structure in  $K^\delta$ ,  $\forall \delta > 0$ , which will allow us to show that, under a particular assumption,  $|K^\delta|$  is large  $\forall \delta > 0$ , and thus  $\dim_N K = n$  for any Kakeya set  $K$  in  $\mathbb{R}^n$  (of course, under the assumption!):

Prop: If :

For all  $\delta > 0$ , for any family  $\mathbb{T}$  of  $\delta$ -separated  $\delta$ -tubes in  $\mathbb{R}^n$ ,

we have that

$$\int_{x \in \mathbb{R}^n} \left( \sum_{T \in \mathbb{T}} \chi_T(x) \right)^{\frac{n}{n-1}} dx \lesssim \delta^{n-1} \# \mathbb{T}$$

↓  
independent  
of  $\delta$

then :

for any Kakeya set  $K$  in  $\mathbb{R}^n$ ,  $|K^\delta| \gtrsim 1$  for all  $\delta > 0$

$$\Rightarrow \dim_M K = n.$$

Maximal Kakeya  
operator conjecture

Proof: Let  $K$  be a Kakeya set in  $\mathbb{R}^n$ , and  $\delta > 0$ .

Let  $\mathbb{T}$  be a family of  $\delta$ -separated  $\delta$ -tubes in  $\mathbb{R}^n$  contained in  $K^\delta$ , with

$\#\mathbb{T} \sim \frac{1}{\delta^{n-1}}$  (we have explained that there exists such a family).

$$\text{Then, } \int_{K^\delta} \sum_{T \in \mathbb{T}} \chi_T(x) dx = \sum_{T \in \mathbb{T}} \int_{K^\delta} \chi_T(x) dx \stackrel{T \subseteq K^\delta \text{ & } T \in \mathbb{T}}{\sim} \sum_{T \in \mathbb{T}} |T| \sim \#\mathbb{T} \cdot \delta^{n-1} \sim \frac{1}{\delta^{n-1}} \cdot \delta^{n-1} \sim 1.$$

any  $\delta$ -tube in  $\mathbb{R}^n$   
has volume = (surface area of its  
cross-section).

On the other hand, by Hölder's inequality:

$$\int_{K^\delta} \sum_{T \in \mathbb{T}} \chi_T \leq \left[ \int_{K^\delta} \left( \sum_{T \in \mathbb{T}} \chi_T \right)^{\frac{n}{n-1}} \right]^{\frac{n-1}{n}} \cdot |K^\delta|^{\frac{1}{n}} \stackrel{\text{by assumption}}{\leq} \left( \delta^{n-1} \#\mathbb{T} \right)^{\frac{n-1}{n}} \cdot |K^\delta|^{\frac{1}{n}} \approx 1^{\frac{n-1}{n}} \cdot |K^\delta|^{\frac{1}{n}} \approx |K^\delta|^{\frac{1}{n}}.$$

$$\text{So, } 1 \lesssim |K^\delta|^{\frac{1}{n}} \Rightarrow |K^\delta| \gtrsim 1, \ \forall \delta > 0.$$

↓  
independent of  $\delta$ .

Therefore,  $\dim_M K = n$ .

We have therefore shown that

Maximal Kakeya  
operator conjecture  $\implies$  Kakeya set conjecture (weak version).

In fact, it can be shown (though it is slightly harder) that

Maximal Kakeya  
operator conjecture  $\implies$  Kakeya set conjecture.

There is a lot of research today on the Maximal Kakeya operator conjecture.

In fact, changing the exponent on the LHS of the conjecture from  $\frac{n}{n-1}$

to some  $q > \frac{n}{n-1}$ , and changing appropriately the exponent of  $\delta$  on the

RHS, we get a worse version of the conjecture, which implies that

any Kakeya set has Hausdorff (and Minkowski) dimension  $\geq q'$  (see your exercises).

[Note that  $n = \left(\frac{n}{n-1}\right)'$ , and for  $q \geq \frac{n}{n-1}$ ,  $q'$  is  $\leq n$ ].

Today we know that the Maximal Kakeya operator conjecture holds for  $q = \frac{n+2}{n}$ ,

so any Kakeya set has Hausdorff dimension  $\geq \left(\frac{n+2}{n}\right)' = \frac{n+2}{2}$ . This is a

result by Wolff (the hairbrush argument, if you are interested!).

## Lecture 22 (11/12/2014)

In the previous lecture, we introduced the following conjecture:

Maximal Kakeya operator conjecture: for all  $\delta > 0$ , for any  $\delta$ -separated family of  $\delta$ -tubes in  $\mathbb{R}^n$ , it holds that

$$\int_{\mathbb{R}^n} \left( \sum_{T \in T} \chi_T \right)^{\frac{n}{n-1}} \lesssim \delta^{n-1} \# T.$$

( $\lesssim$  means  $\leq C_{n,1} \cdot \left( \log \frac{1}{\delta} \right)^{C_{n,2}}$ , for constants  $C_{n,1}, C_{n,2} > 0$ , depending only on  $n$ ).

We have explained that the truth of the maximal Kakeya operator conjecture would imply the truth of the Kakeya set conjecture. Here, we will explain

a little why the maximal Kakeya operator conjecture makes sense.

first of all, we notice that

$$\delta^{n-1} \# \mathbb{T} \sim \int_{\mathbb{R}^n} \sum_{T \in \mathbb{T}} \chi_T .$$

Indeed,

$$\int_{\mathbb{R}^n} \sum_{T \in \mathbb{T}} \chi_T = \sum_{T \in \mathbb{T}} \int_{\mathbb{R}^n} \chi_T = \sum_{T \in \mathbb{T}} |T| \sim \# \mathbb{T} \cdot \delta^{n-1} .$$

And always

$$\int_{\mathbb{R}^n} \left( \sum_{T \in \mathbb{T}} \chi_T \right)^{\frac{n}{n-1}} \geq \int_{\mathbb{R}^n} \sum_{T \in \mathbb{T}} \chi_T , \text{ because } \frac{n}{n-1} \geq 1 \text{ and}$$

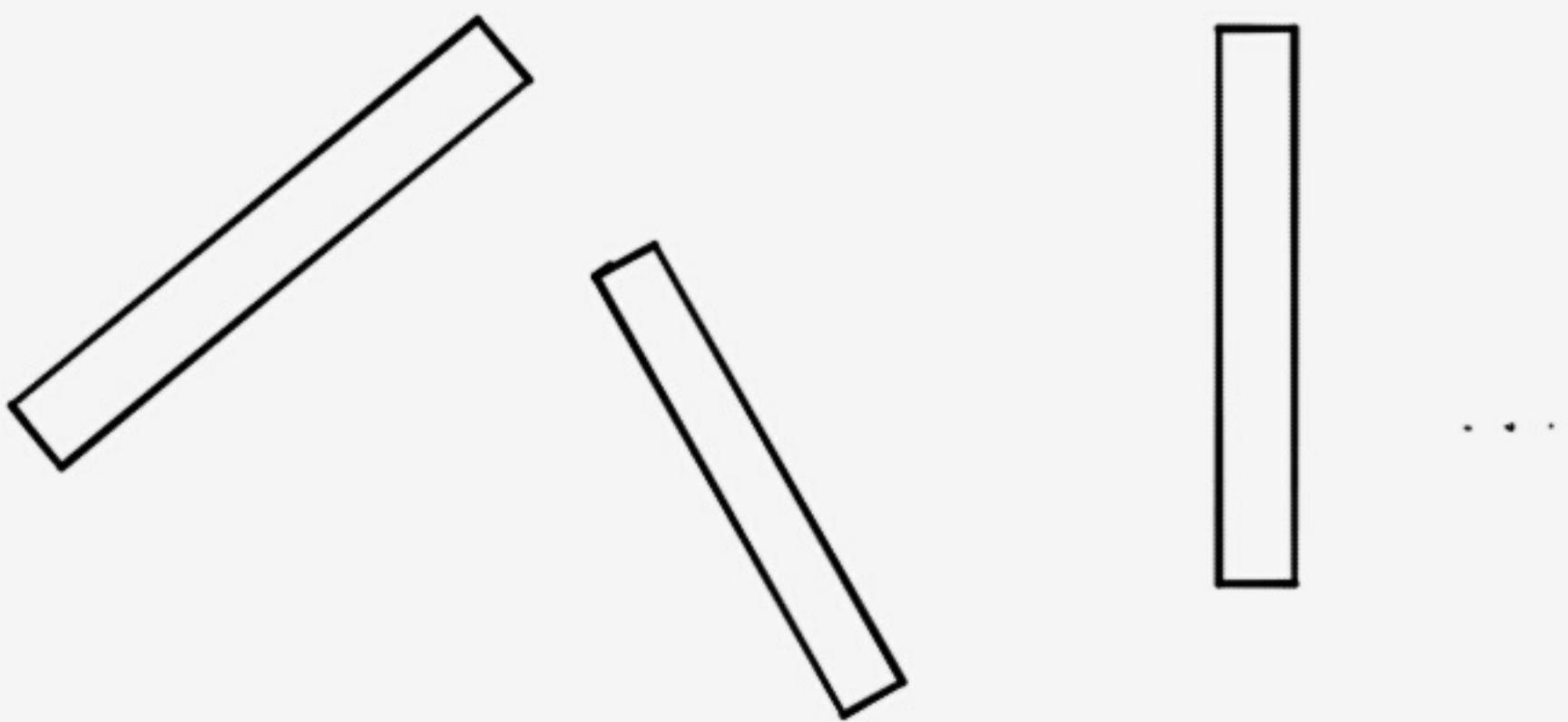
$$\sum_{T \in \mathbb{T}} \chi_T(x) \in \mathbb{N}, \forall x \in \mathbb{R}^n . \text{ Therefore:}$$

$$\int_{\mathbb{R}^n} \left( \sum_{T \in \Pi} \chi_T \right)^{\frac{n}{n-1}} \geq \delta^{n-1} \# \Pi$$

The maximal Kakeya operator conjecture claims that the inverse inequality holds too, as long as one is willing to increase  $\delta^{n-1} \# \Pi$  by a factor of the form  $\left(\log \frac{1}{\delta}\right)^{C_n}$ .

The maximal Kakeya operator conjecture holds in two extremal cases: when all the tubes in  $\Pi$  are pairwise disjoint, and when all the tubes in  $\Pi$  pass through the same point. Indeed:

Case 1:  $\mathbb{T}$  is a  $\delta$ -separated family of  $\delta$ -tubes in  $\mathbb{R}^n$  that are pairwise disjoint:



Since  $\left(\sum_{T \in \mathbb{T}} \chi_T\right)^{\frac{n}{n-1}}$  is 0 outside the union of the tubes, and since the tubes in  $\mathbb{T}$  are pairwise disjoint, it follows that

$$\int_{x \in \mathbb{R}^n} \left( \sum_{T \in \mathbb{T}} \chi_T(x) \right)^{\frac{n}{n-1}} dx = \sum_{T \in \mathbb{T}} \int_{x \in T} \left( \sum_{T \in \mathbb{T}} \chi_T(x) \right)^{\frac{n}{n-1}} dx =$$

$$= \sum_{T \in \mathbb{T}} \int_{x \in T} \# \{ \text{tubes in } \mathbb{T} \text{ through } x \}^{\frac{n}{n-1}} dx \quad \overline{\qquad} \quad \begin{matrix} \hookrightarrow \text{for each } x \in T, \text{ there} \\ \text{exists only 1 tube in } \mathbb{T} \\ \text{passing through } x : T \text{ itself.} \end{matrix}$$

$$= \sum_{T \in \mathbb{T}} \int_{x \in T} 1 dx = \sum_{T \in \mathbb{T}} |T| \sim \#\mathbb{T} \cdot \delta^{n-1}.$$

So, in this case,  $\int_{\mathbb{R}^n} \left( \sum_{T \in \mathbb{T}} \chi_T \right)^{\frac{n}{n-1}} \sim \delta^{n-1} \#\mathbb{T}$  : so, by what we have

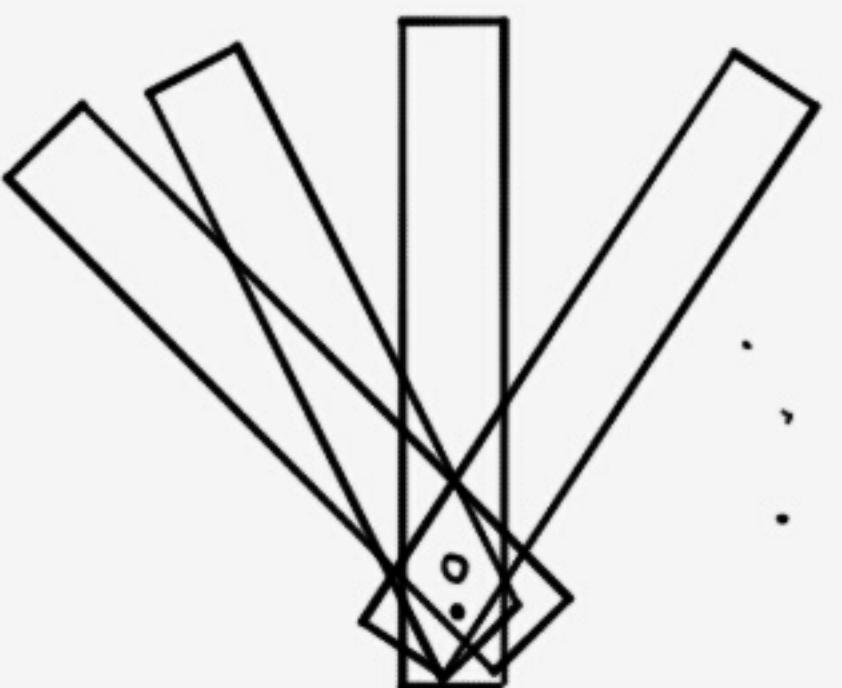
explained earlier, the LHS of the maximal Kakeya operator conjecture is the smallest possible. [Note that another way to see this is the following: here,

$$\sum_{T \in \mathbb{T}} \chi_T \text{ takes only the values 0 and 1, so } \left( \sum_{T \in \mathbb{T}} \chi_T \right)^{\frac{n}{n-1}} = \sum_{T \in \mathbb{T}} \chi_T, \text{ so}$$

$$\int_{\mathbb{R}^n} \left( \sum_{T \in \mathbb{T}} \chi_T \right)^{\frac{n}{n-1}} = \int_{\mathbb{R}^n} \sum_{T \in \mathbb{T}} \chi_T \sim \delta^{n-1} \#\mathbb{T} ].$$

Note that in this case we didn't even need to use the separation of the tubes.

Case 2:  $\mathcal{T}$  is a  $\delta$ -separated family of  $\delta$ -tubes in  $\mathbb{R}^n$ , all passing through the same point (this is known as a bush): say,



We can assume that  $o$  is the root of this bush, i.e. that all the tubes in  $\mathcal{T}$  pass through  $o$ .

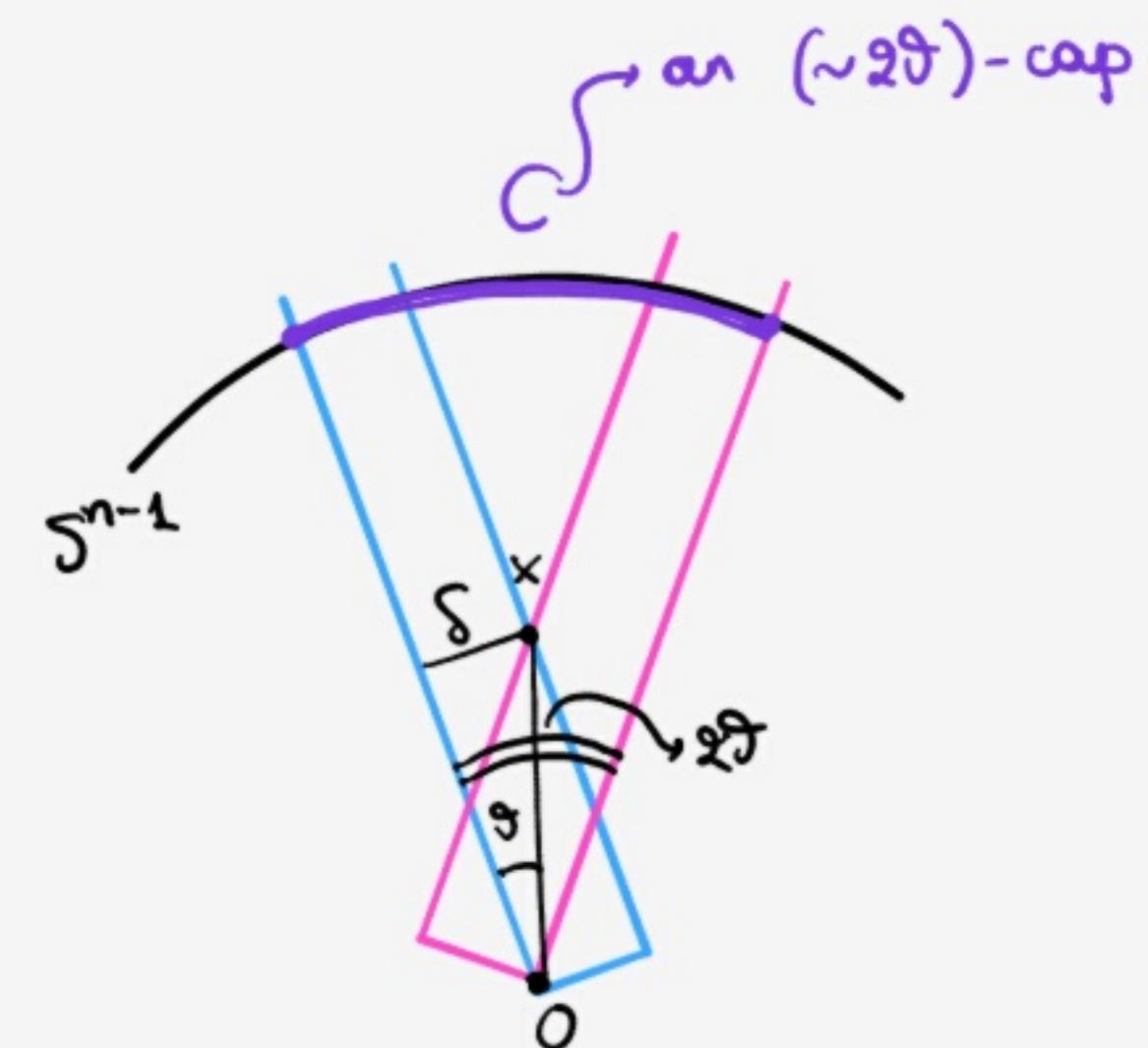
In Case 1, we took advantage of the fact that the tubes were pairwise disjoint. Here they are clearly not: however, because the angle between any two of our tubes is  $\geq \delta$ , the further we are away from the origin, the

"more disjoint" our tubes become! This is what we will use in this case.

It is formally expressed in the following statement:

let  $x \in \mathbb{R}^n$  be a point at distance  $r$  from the root of the bush. Then,  
 $\lesssim \frac{1}{r^{n-1}}$  tubes in  $T$  can pass through  $x$ .

Indeed: Two tubes in  $T$  that pass through  $x$   
also pass through  $0$ . On the right is  
a picture of two such tubes in extremal  
positions. If a tube passes through  $0$   
and  $x$ , then its direction clearly creates  
angle  $\leq \theta$  with the line connecting  $0$  and  $x$ ,



where  $\theta$  is as in the picture. Therefore, the directions of the tubes in  $T$  passing through  $x$  all lie in the cap  $C$ . Moreover, these directions are  $c_n \delta$ -separated on  $S^{n-1}$  (for an appropriate constant  $c_n$  depending only on  $n$ ), since the tubes in  $T$  are  $\delta$ -separated. So:

$$\begin{aligned} \#\{\text{tubes in } T \text{ through } x\} &\leq \#\{c_n \delta\text{-separated directions we can fit in the cap } C\} = \\ &= \#\{\text{pairwise disjoint } \frac{c_n \delta}{2}\text{-caps we can fit in the cap } C\} \sim \end{aligned}$$

$$\sim \frac{\text{surface area of cap } C}{\text{surface area of each } \frac{c_n \delta}{2}\text{-cap}} \sim \frac{(\text{radius of } C)^{n-1}}{\left(\frac{c_n \delta}{2}\right)^{n-1}} \sim \frac{\delta^{n-1}}{\left(\frac{c_n \delta}{2}\right)^{n-1}} \sim \frac{\delta^{n-1}}{\delta^{n-1}}.$$

So, all we need to do is to find  $\delta$ .

Now, from the picture we can see that  $\sin\theta = \frac{\delta}{\text{dist}(x, 0)} = \frac{\delta}{r}$ .

Moreover,  $\theta \sim \sin\theta$ , because

$$\frac{\sin x}{x} \xrightarrow[x \rightarrow 0]{} 1, \text{ so } \sin x \sim x \text{ for } x \text{ small}$$

(and  $\theta$  is small).

$$\text{So, } \theta \sim \frac{\delta}{r}.$$

$$\text{So, } \#\{\text{tubes in } \Pi \text{ through } x\} \lesssim \frac{\left(\frac{\delta}{r}\right)^{n-1}}{\delta^{n-1}} \sim \frac{1}{r^{n-2}}.$$

$$\text{We now proceed by showing that } \int_{\mathbb{R}^n} \left( \sum_{T \in \Pi} \chi_T \right)^{\frac{n}{n-1}} \lesssim \delta^{n-1} \# \Pi.$$

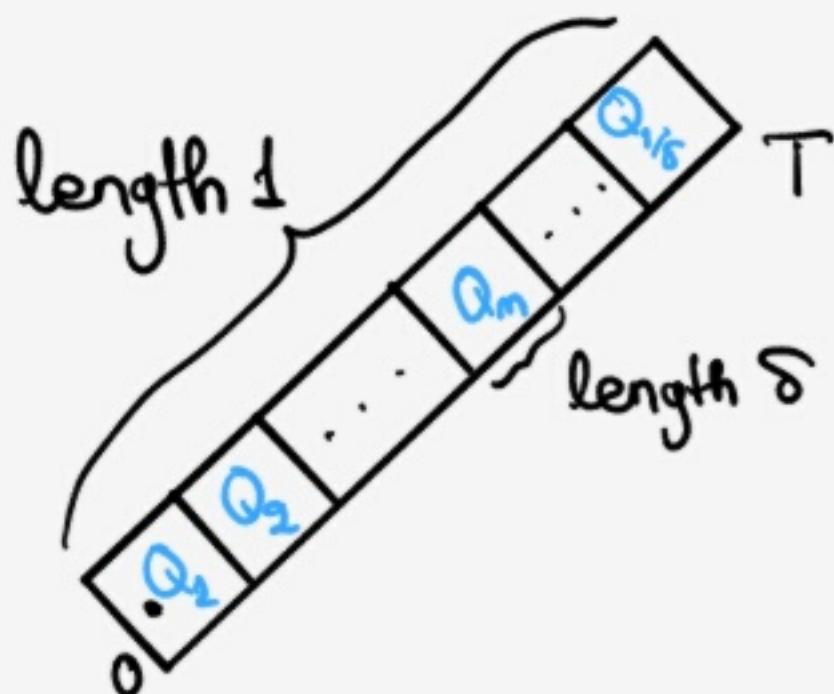
Indeed,

$$\int_{\mathbb{R}^n} \left( \sum_{T \in \Pi} \chi_T \right)^{\frac{n}{n-1}} = \int_{\mathbb{R}^n} \left( \sum_{T \in \Pi} \chi_T \right)^{\frac{1}{n-1}} \cdot \left( \sum_{T \in \Pi} \chi_T \right) = \sum_{T \in \Pi} \int_{\mathbb{R}^n} \left( \sum_{T \in \Pi} \chi_T \right)^{\frac{1}{n-1}} \chi_T =$$

$$= \sum_{T \in \mathbb{T}} \int_{x \in T} \left( \sum_{T' \in \mathbb{T}} \chi_{T'}(x) \right)^{\frac{1}{n-1}} dx =$$

$$= \sum_{T \in \mathbb{T}} \int_{x \in T} \#\{ \text{tubes in } \mathbb{T} \text{ through } x \}^{\frac{1}{n-1}} dx.$$

So, fix  $T \in \mathbb{T}$ . We split  $T$  in  $\sim \frac{1}{\delta}$  pairwise disjoint  $\delta$ -cubes  $Q_1, Q_2, \dots, Q_{1/\delta}$ .



$$\text{Now, } \int_{x \in T} \#\{ \text{tubes in } \mathbb{T} \text{ through } x \}^{\frac{1}{n-1}} dx =$$

$$= \sum_{m=1}^{\frac{1}{\delta}} \int_{x \in Q_m} \#\{ \text{tubes in } \mathbb{T} \text{ through } x \}^{\frac{1}{n-1}} dx. (*)$$

for each  $m$ , if  $x \in Q_m$ , then  $\text{dist}(x, 0) \sim m\delta$ , so

$$\#\{ \text{tubes in } \mathbb{T} \text{ through } x \} \lesssim \frac{L}{(m\delta)^{n-1}}$$

$$\text{Therefore, } (*) \approx \sum_{m=1}^{1/\delta} \int_{x \in Q_m} \left( \frac{1}{(m\delta)^{n-1}} \right)^{\frac{1}{n-1}} dx \sim \sum_{m=1}^{1/\delta} \int_{x \in Q_m} \frac{1}{m\delta} dx \sim$$

$$\sim \sum_{m=1}^{1/\delta} |Q_m| \cdot \frac{1}{m\delta} \sim \sum_{m=1}^{1/\delta} \delta^n \cdot \frac{1}{m\delta} \sim \delta^{n-1} \underbrace{\sum_{m=1}^{1/\delta} \frac{1}{m}}_{\substack{\parallel \\ ?}}.$$

$|Q_m| \sim \delta^n$

Now, for each  $N \in \mathbb{N}$ ,  $\sum_{m=1}^N \frac{1}{m} \sim \log N$ , because, by the definition of an

integral,  $\sum_{m=1}^N \frac{1}{m} \sim \int_1^N \frac{1}{x} dx \sim [\log x]_1^N \sim \log N - \log 1 \sim \log N - 0 \sim \log N$ .

Therefore,  $\delta^{n-1} \sum_{m=1}^{1/\delta} \frac{1}{m} \sim \delta^{n-1} \cdot \log \frac{1}{\delta}$ . So:

$$\int \left( \sum_{T \in \mathbb{T}} \pi_T \right)^{\frac{n}{n-1}} = \sum_{T \in \mathbb{T}} \delta^{n-1} \cdot \log \frac{1}{\delta} \sim \underbrace{\log \frac{1}{\delta}}_{\downarrow} \cdot \delta^{n-1} \# \mathbb{T} \lesssim \delta^{n-1} \# \mathbb{T}.$$

note that this factor cannot be avoided here for general  $\mathbb{T}$  forming a bush!

Note that, in this case, we used the fact that our tubes were  $\delta$ -separated (to force a "disjointness" far from the root).

The maximal Kakeya operator conjecture has so far only been shown to be true for  $n=2$  (which is why we know that any Kakeya set in  $\mathbb{R}^2$  has Hausdorff- and Minkowski-dimension 2.)